

A note on the prevalent dimensions of continuous images of compact spaces

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Abstract

We consider the Banach space consisting of real-valued continuous functions on an arbitrary compact metric space. We prove that for a prevalent set of functions in this space, the Hausdorff and packing dimensions of the image is as large as possible, namely 1. We then use this fact to obtain results on the prevalent dimensions of graphs of real-valued continuous functions on compact metric spaces. A particular case complements a result of Bayart and Heurteaux.

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1 Introduction

Let X be an uncountable compact metric space and let $C(X)$ denote the set of real-valued continuous functions on X . $C(X)$ is a Banach space when equipped with the supremum norm, $\|\cdot\|_\infty$. We investigate the Hausdorff and packing dimensions of three objects related to $f \in C(X)$, namely, the image

$$f(X) \subseteq \mathbb{R},$$

the product space

$$X \times f(X) \subseteq X \times \mathbb{R}$$

and the graph

$$G_f \equiv \{(x, f(x)) : x \in X\} \subset X \times f(X).$$

Rather than compute these dimensions for specific examples, we look to find the ‘generic answer’. Namely, what are the dimensions of the above sets for a generic $f \in C(X)$? In order to do this we need a suitable notion of genericity in Banach spaces, which we obtain using the theory of *prevalence*, see Section 1.2 for an account of the theory.

Prevalence has been used extensively in the literature to study dimensional properties of generic continuous functions. In particular, over the past 15 years there has been considerable interest in studying the prevalent dimensions of graphs of continuous functions, see [BH, FF, FH, GJMNO, Mc, S]. The most general result to date has been given by Bayart and Heurteaux [BH].

Theorem 1.1. *Let X be a compact subset of \mathbb{R}^d with positive Hausdorff dimension. The set*

$$\{f \in C(X) : \dim_{\text{H}} G_f = \dim_{\text{H}} X + 1\}$$

is a prevalent subset of $C(X)$.

The case where $X = [0, 1]^d$ was proven by the authors [FH]. The method of proof used in [BH] was to use fractional Brownian motion on X . The assumption that X has positive Hausdorff dimension was needed to guarantee the existence of an appropriate measure to use in the energy estimates. Interestingly, this left the case where $\dim_H X = 0$ to consider. Clearly if X is finite or countable then the dimension of the graph is necessarily 0, so the only open case is when X has cardinality continuum but is zero dimensional. We will solve this problem in the following section, see Corollary 2.3 and Theorem 2.4.

There has also been interest in using prevalence to study the generic dimension of images of continuous functions. This problem is related to the seminal results of Mattila [Ma], Kaufman, [K] and Marstrand [M] on the almost sure dimension of orthogonal projections of sets in Euclidean spaces. For example, Marstrand's projection theorem states that if $F \subset \mathbb{R}^2$ is Borel, then for almost all linear subspaces of the plane, the Hausdorff dimension of the corresponding orthogonal projection is equal to $\min\{1, \dim_H F\}$, i.e., the dimension is generically preserved. In [SY], prevalence was used to extend the results on projections to the space of all continuously differentiable maps. Again it was found that the dimension is generically preserved, this time using prevalence to give a notion of 'generic'. In this paper we show that if one considers the much larger space of merely continuous functions, then the Hausdorff dimension is not preserved and in fact it is 'almost surely' as large as possible, see Theorem 2.1.

1.1 Dimension

In this paper we will be concerned with the Hausdorff and packing dimension of the three sets related to $f \in C(X)$ described above. For a subset F of a metric space (X, d) we will write $\dim_H F$ and $\dim_P F$ for the Hausdorff and packing dimensions of F respectively. For definitions and basic properties see [F1]. In particular, we will use the following proposition.

Proposition 1.2. *Let (X, d_X) and (Y, d_Y) be arbitrary metric spaces. Then*

$$\dim_H X + \dim_H Y \leq \dim_H X \times Y \leq \dim_H X + \dim_P Y \leq \dim_P X \times Y \leq \dim_P X + \dim_P Y$$

where we use the product metric $d_{X \times Y}$ on $X \times Y$ defined by

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Furthermore, if X, Y are Euclidean spaces then the above relationships remain true when we use the Euclidean metric on the product space instead of the product metric.

For a proof of this see [H].

1.2 Prevalence

'Prevalence' provides one way of describing the *generic* behavior of a class of mathematical objects. In finite dimensional vector spaces Lebesgue measure provides a natural tool for deciding if a property is 'generic'. Namely, if the set of elements which do not have some property is a Lebesgue null set then it is said that this property is 'generic' from a measure theoretical point of view. However, when the space in question is infinite dimensional this approach breaks down because there is no useful analogue to Lebesgue measure in the infinite dimensional setting. The theory of prevalence has been developed to solve this problem. It was first introduced in the general setting of abelian Polish groups by Christensen in the 1970s [Ch, Ch2] and later rediscovered by Hunt, Sauer and Yorke in 1992 [HSY]. We will now give a brief reminder of the definitions we will need.

Definition 1.3. *A completely metrizable topological vector space is a vector space, X , for which there exists a metric, d , on X such that (X, d) is complete and the vector space operations are continuous with respect to the topology induced by d .*

Note that $(C(X), \|\cdot\|_\infty)$ is a completely metrizable topological vector space with the topology induced by the norm. Also, it is infinite dimensional and so it is natural to appeal to the theory of prevalence.

Definition 1.4. *Let X be a completely metrizable topological vector space. A Borel set $F \subseteq X$ is prevalent if there exists a Borel measure μ on X and a compact set $K \subseteq X$ such that $0 < \mu(K) < \infty$ and*

$$\mu(X \setminus (F + x)) = 0$$

for all $x \in X$.

The complement of a prevalent set is called a shy set and a non-Borel set $F \subseteq X$ is prevalent if it contains a prevalent Borel set.

The measure μ mentioned in the above definition can often be taken to be Lebesgue measure concentrated on a finite dimensional subset of X .

Definition 1.5. A k -dimensional subspace $P \subseteq X$ is called a probe for a Borel set $F \subseteq X$ if

$$\mathcal{L}_P(X \setminus (F + x)) = 0$$

for all $x \in X$ where \mathcal{L}_P denotes the natural pull back of k -dimensional Lebesgue measure to P . We call F k -prevalent if it admits a k -dimensional probe.

The existence of a probe is clearly a sufficient condition for a set F to be prevalent and being k -prevalent is a stronger condition than being prevalent. Shyness is a reasonable generalisation of Lebesgue measure zero to the infinite dimensional setting. It enjoys many of the natural properties which one would expect from such a generalisation, for example, it is preserved under taking countable unions, and, in particular, in \mathbb{R}^d being shy is equivalent to having Lebesgue measure zero. For more details see [OY].

2 Results

Our main result is the following.

Theorem 2.1. The set of functions $f \in C(X)$ for which $\dim_H f(X) = \dim_P f(X) = 1$ is a 1-prevalent set.

The proof of this result is deferred to Section 3.2. We can use Theorem 2.1 and Proposition 1.2 to glean information about the product space and the graph of a prevalent continuous function.

Corollary 2.2. The set of functions $f \in C(X)$ for which

$$\begin{aligned} \dim_H (X \times f(X)) &= \dim_H X + 1, \\ \dim_P (X \times f(X)) &= \dim_P X + 1, \\ \max\{1, \dim_H X\} &\leq \dim_H G_f \leq \dim_H X + 1 \end{aligned}$$

and

$$\max\{1, \dim_P X\} \leq \dim_P G_f \leq \dim_P X + 1$$

is a 1-prevalent set.

Corollary 2.2 follows immediately from Theorem 2.1 and Proposition 1.2. A further specialisation gives the following.

Corollary 2.3. Suppose $\dim_H X = 0$. Then the set of functions $f \in C(X)$ for which

$$\dim_H G_f = 1$$

is a 1-prevalent set. If we also have $\dim_P X = 0$, then the set of functions $f \in C(X)$ for which

$$\dim_H G_f = \dim_P G_f = 1$$

is a 1-prevalent set.

A combination of Corollary 2.3 and the result of Bayart and Heurteaux gives.

Theorem 2.4. Let X be an uncountable compact subset of \mathbb{R}^d . The set

$$\{f \in C(X) : \dim_H G_f = \dim_H X + 1\}$$

is a prevalent subset of $C(X)$. If X is finite or countable, then $\dim_H G_f = 0$ for all $f \in C(X)$.

We remark here that a compact subset of \mathbb{R}^d is either finite, countable or has cardinality continuum (see [C], Corollary 6.2.5) and so we do not need to consider the continuum hypothesis in the above Theorem. It is interesting that in the case where X is zero dimensional the above result can be obtained using a probe, but when the dimension of X is strictly positive it is unclear if a probe can be used. In particular, the proofs in [BH, FH] do not use a probe. We therefore ask the following question.

Question 2.5. Does Theorem 2.4 remain true if prevalent is replaced by 1-prevalent?

3 Proofs

3.1 Potential theoretic approach

Our key tool in estimating Hausdorff dimension will be potential theory. In this section we will briefly recall the key definitions and properties. Potential theoretic methods provide a powerful tool for finding lower bounds for Hausdorff dimension. Let $s \geq 0$ and let μ be a probability measure on \mathbb{R}^d . The s -energy of μ is defined by

$$I_s(\mu) = \iint \frac{d\mu(x) d\mu(y)}{|x - y|^s}.$$

The following theorem relates the Hausdorff dimension of a set, F , to the s -energy of probability measures supported on F . We will write $\mathcal{H}^s(F)$ for the Hausdorff measure of F

Proposition 3.1. *Let $F \subset \mathbb{R}^d$ be a Borel set. If there exists a Borel probability measure μ on F with $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and therefore $\dim_H F \geq s$. Conversely, if $\mathcal{H}^s(F) > 0$, then there exists a Borel probability measure μ on F with $I_t(\mu) < \infty$ for all $0 < t < s$.*

For a proof of this result see [F1], Theorem 4.13.

3.2 Proof of Theorem 2.1

In this section we will prove Theorem 2.1. The following Lemma is our key technical result.

Lemma 3.2. *Let $\phi \in C(X)$ be an injection and let $f \in C(X)$ be arbitrary. For almost all $\lambda \in \mathbb{R}$ we have*

$$\dim_H(f + \lambda\phi)(X) \geq \dim_H \phi(X).$$

Proof. Let $\phi \in C(X)$ be an injection and write $s = \dim_H \phi(X) \leq 1$. Also fix $f \in C(X)$ and $\varepsilon \in (0, s)$. It follows from Proposition 3.1 that there exists a Borel probability measure μ supported on $\phi(X)$ with finite $s - \varepsilon$ energy, i.e.,

$$I_{s-\varepsilon}(\mu) < \infty. \tag{3.1}$$

Since ϕ is a bijection between X and $\phi(X)$ we can define a pull back measure $\nu = \mu \circ \phi$ on X . Finally, for $\lambda \in \mathbb{R}$, we let $\nu_\lambda = \nu \circ (f + \lambda\phi)^{-1}$ be the image measure of ν under $f + \lambda\phi$. Since ν_λ is supported on $(f + \lambda\phi)(X)$, it suffices to show that for almost all $\lambda \in \mathbb{R}$ the measure ν_λ has finite $(s - \varepsilon)$ -energy, i.e.,

$$I_{s-\varepsilon}(\nu_\lambda) < \infty.$$

For $\lambda \in \mathbb{R}$ we have the following expression for $I_{s-\varepsilon}(\nu_\lambda)$.

$$\begin{aligned} I_{s-\varepsilon}(\nu_{\lambda,f}) &= \int_{x \in f(K)} \int_{y \in f(K)} \frac{d\nu_\lambda(x) d\nu_\lambda(y)}{|x - y|^{s-\varepsilon}} \\ &= \int_{x \in f(K)} \int_{y \in f(K)} \frac{d(\nu \circ (f + \lambda\phi)^{-1})(x) d(\nu \circ (f + \lambda\phi)^{-1})(y)}{|(f + \lambda\phi) \circ (f + \lambda\phi)^{-1}(x) - (f + \lambda\phi) \circ (f + \lambda\phi)^{-1}(y)|^{s-\varepsilon}} \\ &= \int_{u \in K} \int_{v \in K} \frac{d\nu(u) d\nu(v)}{|(f + \lambda\phi)(u) - (f + \lambda\phi)(v)|^{s-\varepsilon}} \\ &= \int_{u \in K} \int_{v \in K} \frac{d\nu(u) d\nu(v)}{|(f(u) - f(v)) + \lambda(\phi(u) - \phi(v))|^{s-\varepsilon}} \end{aligned} \tag{3.2}$$

It follows from (3.2) and Fubini's Theorem that

$$\begin{aligned}
\int_{-n}^n \left(I_{s-\varepsilon}(\nu_\lambda) \right) d\lambda &= \int_{-n}^n \int_{u \in K} \int_{v \in K} \frac{d\nu(u) d\nu(v)}{|(f(u) - f(v)) + \lambda(\phi(u) - \phi(v))|^{s-\varepsilon}} d\lambda \\
&= \int_{u \in K} \int_{v \in K} \int_{-n}^n \frac{d\lambda}{|(f(u) - f(v)) + \lambda(\phi(u) - \phi(v))|^{s-\varepsilon}} d\nu(u) d\nu(v) \\
&\leq \int_{u \in K} \int_{v \in K} 2 \int_0^{2n} \frac{d\lambda}{\lambda^{s-\varepsilon} |\phi(u) - \phi(v)|^{s-\varepsilon}} d\nu(u) d\nu(v) \\
&= 2 \frac{(2n)^{1-s+\varepsilon}}{1-s+\varepsilon} \int_{u \in K} \int_{v \in K} \frac{d\nu(u) d\nu(v)}{|\phi(u) - \phi(v)|^{s-\varepsilon}} \\
&\leq \frac{8n^2}{1-s+\varepsilon} \int_{u \in K} \int_{v \in K} \frac{d(\mu \circ \phi)(u) d(\mu \circ \phi)(v)}{|\phi(u) - \phi(v)|^{s-\varepsilon}} \\
&= \frac{8n^2}{1-s+\varepsilon} \int_{p \in \phi(K)} \int_{q \in \phi(K)} \frac{d\mu(p) d\mu(q)}{|p - q|^{s-\varepsilon}} \\
&= \frac{8n^2}{1-s+\varepsilon} I_{s-\varepsilon}(\mu) \\
&< \infty
\end{aligned}$$

by (3.1). It follows that $\dim_{\mathbb{H}}(f + \lambda\phi)(K) \geq s - \varepsilon$ for almost all $\lambda \in (-n, n)$ and hence $\dim_{\mathbb{H}}(f + \lambda\phi)(K) \geq \dim_{\mathbb{H}} \phi(K)$ for almost all $\lambda \in \mathbb{R}$. Note that we may apply Fubini's Theorem above because the measures involved are finite and the final integral is finite. \square

The proof of Theorem 2.1 now follows easily. We begin with a technical lemma.

Lemma 3.3. *The set*

$$\{f \in C(K) : \dim_{\mathbb{H}} f(K) = 1\}$$

is a Borel subset of $C(K)$.

Proof. Let $\mathcal{K}(\mathbb{R})$ denote the set consisting of all non-empty compact subsets of \mathbb{R} and equip this space with the Hausdorff metric, $d_{\mathcal{H}}$. It was shown in [MM] that the function $\Delta_{\mathbb{H}} : (\mathcal{K}(\mathbb{R}), d_{\mathcal{H}}) \rightarrow \mathbb{R}$ defined by

$$\Delta_{\mathbb{H}}(K) = \dim_{\mathbb{H}} K$$

is of Baire class 2, and, in particular, Borel measurable. Let $\Lambda : C(K) \rightarrow \mathbb{R}$ be defined by $\Lambda(f) = f(X)$ and observe that it is continuous. It follows that

$$(\Delta_{\mathbb{H}} \circ \Lambda)^{-1}(\{1\}) = \{f \in C(K) : \dim_{\mathbb{H}} f(K) = 1\}$$

is a Borel set. \square

We can now prove Theorem 2.1.

Proof. Since X is compact it follows that X contains a closed subset, K , homeomorphic to the Cantor space, see [BBT], Theorem 11.11. Let $F \subset [0, 1]$ be a Cantor set with Hausdorff dimension equal to 1. It is clear that such a set exists and for an explicit construction see the ‘fat Cantor set’ constructed in [FH]. Since all Cantor sets are homeomorphic there exists a continuous bijection ϕ mapping K to F . We can use Tietze's Extension Theorem to extend ϕ to a continuous function $\Phi \in C(X)$. We claim that

$$P := \{\lambda\Phi : \lambda \in \mathbb{R}\}$$

is a probe for $A := \{f \in C(X) : \dim_{\mathbb{H}} f(X) = 1\}$. Let $\pi_P : P \rightarrow \mathbb{R}$ be defined by $\pi_P(\lambda\phi) = \lambda$ and let $\mathcal{L}_P = \mathcal{L}^1 \circ \pi_P$ be 1-dimensional Lebesgue measure on P . Fix $f \in C(X)$. We have

$$\begin{aligned}
\mathcal{L}_P(C(X) \setminus (A + f)) &= \mathcal{L}_P(P \setminus (A + f)) \\
&= (\mathcal{L}^1 \circ \pi_P)(\lambda\phi : \lambda\phi - f \notin A) \\
&= \mathcal{L}^1(\lambda : \dim_{\mathbb{H}}(-f + \lambda\phi)(X) < 1) \\
&\leq \mathcal{L}^1(\lambda : \dim_{\mathbb{H}}(-f + \lambda\phi)(K) < \dim_{\mathbb{H}} \phi(K)) \\
&= 0
\end{aligned}$$

by Lemma 3.2. Theorem 2.1 now follows from Lemma 3.3 and the fact that A admits a 1-dimensional probe. \square

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